A note on stochastic semilinear equations and their associated Fokker-Planck equations *

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Abstract

In this paper we treat semilinear stochastic partial differential equations by two methods. First, we extend the framework of [BDR10] from a Hilbert space to a Gelfand triple and as an application we prove the existence of solutions for the Fokker-Planck equations associated to semilinear equations with space-time white noise and both with polynomially growing nonlinearities and Burgers type nonlinearities at the same time. Second we adopt the approximation technique from [BDR10] to obtain existence of unique strong solutions to semilinear stochastic partial differential equations driven by space-time white noise, generalizing corresponding known results from the literature.

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1 Introduction

We consider the semilinear stochastic partial differential equation (SPDE)

$$dX(t) = (\frac{\partial^2}{\partial \xi^2}X(t) + f(t, X(t)) + \frac{\partial}{\partial \xi}g(t, X(t)))dt + \sqrt{C}dW(t), \tag{1.1}$$

on $H := L^2(0, 1)$ with Dirichlet boundary condition

$$X(t, 0) = X(t, 1) = 0, t \in [0, T],$$

and initial condition

$$X(0) = x \in H.$$

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where $f(t, \xi, r), g(t, \xi, r)$ are Borel measurable functions of $(t, \xi, r) \in \mathbb{R}_+ \times [0, 1] \times \mathbb{R}$, W is a cylindrical Wiener process on H and C is a linear positive definite operator in H.

This kind of stochastic partial differential equations has been studied intensively. If $f = 0$ and $g = \frac{1}{2}r^2$, the above equation is called stochastic Burgers equation and has been investigated in many papers (see e.g. [DDT94], [DZ96] and the references therein). When $g = 0$ then the above equation is a stochastic reaction-diffusion equation which has also attracted a lot of attention (see e.g. [DZ92], [D04], [BDR10] and the references therein). In general, this kind of equations has been studied e.g. in [G98], [GR00], where however, f was assumed to be of linear growth. In the present paper we allow f to grow polynomially of arbitrary order and Burgers type nonlinearities g at the same time. We stress that the linear growth of f can not be dropped in [G98], [GR00], since the approximation technique used there uses this assumption.

Here we use two different approaches to study this equation: namely via Fokker-Planck equations and via martingale problems.

In the first part of this paper we study the associated Fokker-Planck equation corresponding to (1.1). Recently, there has been quite an interest in Fokker-Planck equations with irregular coefficients in finite dimensions (see e.g. [A04], [DPL89], [F08], [BDR08a] and the references therein). In [BDR08b], [BDR09] and [BDR10], Bogachev, Da Prato and the first named author of this paper have started the study of Fokker-Planck equations in infinite dimensions, more precisely, on Hilbert spaces. They prove the existence and uniqueness of solutions for Fokker-Planck equations for the case of full noise (i.e. the diffusion operator is invertible) and for trace class noise under monotonicity conditions on the non-linear part of the drift. In this paper, we extend their result to a more general framework which includes the above class of equations as an application. Here we would like to stress that we can prove the existence of solutions to Fokker-Planck equations for the stochasic Burgers equation plus a reaction diffusion term with polynomial growth of any order. We also emphasize that our noise does not need to be trace-class and we take space-time white noise as an example.

Let us recall some notions and the framework for Fokker-Planck equations. Let H be a separable real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $|\cdot|$. $L(H)$ denotes the set of all bounded linear operators on H, $\mathcal{B}(H)$ its Borel σ-algebra.

Consider the following type of non-autonomous stochastic differential equations on H and time interval $[0, T]$:

$$\begin{cases} dX(t) = (AX(t) + F(t, X(t)))dt + \sqrt{C}dW(t), \\ X(s) = x \in H, t \geq s. \end{cases} \tag{1.2}$$

Here $W(t), t \geq 0$, is a cylindrical Wiener process on H defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, C is a linear positive definite operator in H, $D(F) \in \mathcal{B}([0, T] \times H)$, $F : D(F) \subset [0, T] \times H \to H$ is a Borel measurable map, and $A : D(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup $e^{tA}, t \geq 0$, on H.

The Kolmogorov operator L_0 corresponding to (1.2) reads as follows:

$$L_0 u(t, x) := D_t u(t, x) + \frac{1}{2} Tr[CD^2 u(t, x)] + \langle x, A^* Du(t, x)\rangle + \langle F(t, x), Du(t, x)\rangle, \ (t, x) \in D(F),$$

where D_t denotes the derivative in time and D, D^2 denote the first- and second-order Frechet derivatives in space, i.e., in $x \in H$, respectively. The operator L_0 is defined on the space

$D(L_0) := \mathcal{E}_A([0,T] \times H)$, defined to be the linear span of all real and imaginary parts of all functions $u_{\phi,h}$ of the form

$$u_{\phi,h}(t,x) = \phi(t)e^{i\langle x,h(t)\rangle}, t \in [0,T], x \in H,$$

where $\phi \in C^1([0,T])$, $\phi(T) = 0$, $h \in C^1([0,T]; D(A^*))$ and A^* denotes the adjoint of A.

For a fixed initial time $s \in [0,T]$ the Fokker-Planck equation is an equation for measures $\mu(dt,dx)$ on $[s,T] \times H$ of the type

$$\mu(dt,dx) = \mu_t(dx)dt,$$

with $\mu_t \in \mathcal{P}(H)$ for all $t \in [s,T]$, and $t \mapsto \mu_t(A)$ measurable on $[s,T]$ for all $A \in \mathcal{B}(H)$, i.e. $\mu_t(dx), t \in [s,T]$, is a probability kernel from $([s,T], \mathcal{B}([s,T])$ to $(H, \mathcal{B}(H))$. Then the equation for an initial condition $\zeta \in \mathcal{P}(H)$ reads as follows: $\forall u \in D(L_0)$

$$\int_H u(t,y)\mu_t(dy) = \int_H u(s,y)\zeta(dy) + \int_s^t ds' \int_H L_0 u(s',y)\mu_{s'}(dy), \text{ for } dt - a.e.t \in [s,T], \quad (1.3)$$

where the dt-zero set may depend on u.

In the first part of this paper, we extend the abstract framework of [BDR10] from a Hilbert space to a Gelfand triple (see Section 2) and improve their results. As an application we prove the existence of solutions for the Fokker-Planck equations associated with concrete SPDE of type (1.1) i.e. allowing polynomially growing nonlinearities in the reaction-diffusion part and Burgers type nonlinearities at the same time (see Section 3). This general type of equations could not be handled within the framework of [BDR10]. We stress that we only work in the case where $C^{-1} \in L(H)$, i.e. the case of full (including white) noise. If $TrC < \infty$, there are many known existence results (cf. [BDR08b, BDR09]) based on the method of constructing Lyapunov functions with weakly compact level sets for the Kolmogorov operator L_0, which so far could not be used when $TrC = \infty$.

In the second part of this paper (see Section 4) under a stronger condition on the t-dependence of f (see (4.4) below), we use standard methods and a similar approximation as in Section 3 to prove the existence of martingale solutions for the above concrete semilinear SPDE driven by space-time white noise of type (1.1) (see Theorem 4.1 and Remark 4.2). Moreover, the weak uniqueness of the martingale solution follows from [MR99]. Under an additional condition on f (cf. (4.7) below), we also obtain pathwise uniqueness and by the Yamada-Watanabe theorem we get existence and uniqueness of a (probabilistically) strong solution (Theorem 4.4). Since, as mentioned before, we can include nonlinearities f of polynomial growth of any order and Burgers type nonlinearities g at the same time, we thus generalize the results from [G98].

2 Existence of solutions for Fokker-Planck equations

Let us first introduce some assumptions to be used below.

Hypothesis 2.1 (i) A is self-adjoint and such that there exists $\omega \in \mathbb{R}$ such that $\langle Ax, x\rangle \le \omega|x|^2, x \in D(A)$.

(ii) $C \in L(H)$ is symmetric, nonnegative and such that $C^{-1} \in L(H)$.

(iii) There exists $\delta \in (0, 1/2)$ such that $(-A)^{-2\delta}$ is of trace class.

We change Hypotheses 2.2, 2.3 in [BDR10] as follows: let $V := D((-A)^{1/2})$ and consider the following Gelfand triple:

$$V \subset H \subset V^*,$$

where V^* is the dual of V. Furthermore, we relax the assumptions on F in (1.2) to be just V^*-valued. More precisely, let $F : D(F) \subset [0, T] \times H \to V^*$ be Borel measurable. Then the Kolmogorov operator is given as follows

$$L_0 u(t, x) := D_t u(t, x) + \frac{1}{2} Tr[CD^2 u(t, x)] + \langle x, A^* Du(t, x)\rangle + {}_{V^*}\langle F(t, x), Du(t, x)\rangle_V,$$

for $u \in D(L_0)$.

Hypothesis 2.2 There exist measurable maps $F_\alpha : [0, T] \times D(F) \to V^*, \alpha \in (0, 1]$, $K > 0$ and a lower semicontinuous function $J : [s, t] \times H \to [1, \infty]$, such that the following three conditions are satisfied:

(i) for all $(t, x) \in D(F)$ and all $h \in D(A)$

$$|F_\alpha(t, x)|_{V^*} \leq |F(t, x)|_{V^*},$$

$$|{}_{V^*}\langle F(t, x) - F_\alpha(t, x), h\rangle_V| \leq \alpha c(h) J(t, x),$$

for some constant $c(h) > 0$.

(ii) The following approximating stochastic equations for $\alpha \in (0, 1]$ and fixed $s \in [0, T]$

$$dX_\alpha(t) = [AX_\alpha(t) + F_\alpha(t, X_\alpha(t))]dt + \sqrt{C} dW(t), X_\alpha(s) = x, s \leq t,$$

have a martingale solution in the sense of [DZ92] which we denote by $X_\alpha(\cdot, s, x)$.

(iii)$|F|_{V^*} \leq J$ on $[s, T] \times H$, where we set $|F|_{V^*} := +\infty$ on $[s, T] \times H \backslash D(F)$, and setting

$$P^\alpha_{s,t}\varphi(x) := E[\varphi(X_\alpha(t, s, x))], \quad 0 \leq s < t \leq T, \varphi \in \mathcal{B}_b(H),$$

we have

$$P^\alpha_{s,t} J^2(t, \cdot)(x) \leq K J^2(t, x) < \infty, \forall (t, x) \in D(F), t \in [s, T], \alpha \in (0, 1].$$

Set

$$W_A(t, s) = \int_s^t e^{(t-s')A} \sqrt{C} dW(s'), \quad t \geq s.$$

Fix $s \in [0, T)$ and set

$$\mu^\alpha_t(dx) := (P^\alpha_{s,t})^* \zeta(dx),$$

where $\zeta \in \mathcal{P}(H)$ is the initial condition, at $t = s$.

Theorem 2.3 Assume Hypotheses 2.1, 2.2 and that

$$(t, x) \mapsto {}_{V^*}\langle F_\alpha(t, x), h\rangle_V \text{ is continuous on } [s, T] \times H, \forall h \in D(A), \alpha \in (0, 1].$$

Let $\zeta \in \mathcal{P}(H)$ be such that

$$\int_s^T \int_H (J^2(s',x) + |x|^2)\zeta(dx)ds' < \infty.$$

Then there exists a solution $\mu_t(dx)dt$ to the Fokker-Planck equation (1.3) such that

$$\sup_{t\in[s,T]} \int_H |x|^2 \mu_t(dx) < \infty$$

and

$$t \mapsto \int_H u(t,x)\mu_t(dx)$$

is continuous on $[s,T]$ for all $u \in D(L_0)$. Finally, for some $C > 0$ one has

$$\int_s^T \int_H (J^2(s',x) + |(-A)^\delta x|^2 + |x|^2)\mu_{s'}(dx)ds' \le C \int_s^T \int_H (J^2(s',x) + |x|^2)\zeta(dx)ds'.$$

Proof For $\alpha \in (0,1]$, set $X_\alpha(t) := X_\alpha(s,t,x), x \in H$, and

$$Y_\alpha(t) := X_\alpha(t) - W_A(t,s), \quad t \ge s.$$

Then in the mild sense

$$\frac{d}{dt}Y_\alpha(t) = AY_\alpha(t) + F_\alpha(t, X_\alpha(t)), \quad t > s.$$

Applying $\langle Y_\alpha(t), \cdot\rangle$ to both sides and integrating over $[s,T]$, we obtain

$$|Y_\alpha(t)|^2 + 2\int_s^t |(-A)^{1/2}Y_\alpha(s')|^2 ds' \le |x|^2 + \int_s^t (|(-A)^{1/2}Y_\alpha(s')|^2 + |F_\alpha(s', X_\alpha(s'))|_{V^*}^2)ds'.$$

Taking expectation and applying Hypothesis 2.2 yields

$$E|Y_\alpha(t)|^2 \le |x|^2 + K\int_s^t |J(s',x)|^2 ds', \quad t \ge s.$$

Then for $s \le t \le T$ we obtain

$$E|X_\alpha(t)|^2 \le 2|x|^2 + 2K\int_s^T |J(s',x)|^2 ds' + 2\kappa,$$

where $\kappa := \sup_{t\in[s,T]} E|W_A(t)|^2 < \infty$. Now we integrate with respect to ζ over $x \in H$ and obtain for $s \le t \le T$

$$\int_H |x|^2 \mu_t^\alpha(dx) \le C[1 + \int_s^T \int_H (J(s',x)^2 + |x|^2)\zeta(dx)ds'],$$

for some $C > 0$. By this we can use Prohorov' theorem (see [B07, Theorem 8.6.7]) to obtain that for each $t \in [s,T]$, there exists a sub-sequence $\{\alpha_n\}$ (possibly depending on t) such that

the measures $\mu_t^{\alpha_n}$ converge τ_w-weakly to a measure $\tilde{\mu}_t \in \mathcal{P}(H)$ as $n \to \infty$, where τ_w denotes the weak topology on H. Then by the same arguments as in the proof of [BDR10, Theorem 2.6], we can construct a measure μ_t and a subsequence $\{\alpha_n\}$ such that $\mu_t^{\alpha_n}$ converge τ_w-weakly to μ_t for all $t \in [0,T]$. Now for $\delta \in (0, \frac{1}{2})$ as in Hypothesis 2.1 (iii) we obtain

$$\int_s^T \int_H |(-A)^\delta x|^2 \mu_t^\alpha(dx)dt \le C[1 + \int_s^T \int_H (J(s',x)^2 + |x|^2)\zeta(dx)ds'],$$

which implies that $\mu_t^{\alpha_n}(dx)dt$ converge weakly to $\mu_t(dx)dt$ on $[0,T] \times H$ by the compactness of $(-A)^{-\delta}$. Now we only need to prove that $\mu_t(dx)dt$ solves the Fokker-Planck equation (1.2). It suffices to prove that for all $g \in C_b([s,T] \times H)$ and all piecewise affine $h \in C([0,T]; D(A))$,

$$\lim_{n\to\infty} \int_s^T \int_H F_{\alpha_n}^h(t,x) g(t,x) \mu_t^{\alpha_n}(dx)dt = \int_s^T \int_H F^h(t,x) g(t,x) \mu_t(dx)dt, \tag{2.1}$$

where

$$F_\alpha^h(t,x) := {}_{V^*}\langle F_\alpha(t,x), h(t)\rangle_V + \frac{\langle Ah(t), x\rangle}{1 + \alpha|\langle Ah(t), x\rangle|},$$

$$F^h(t,x) := {}_{V^*}\langle F(t,x), h(t)\rangle_V + \langle Ah(t), x\rangle.$$

By Hypothesis 2.2 we have for all $\alpha, \beta \in (0,1]$

$$\int_s^T \int_H |F_\beta^h(t,x) - F^h(t,x)| \mu_t^\alpha(dx)dt \le \beta\gamma(h) \int_s^T \int_H (|F(t,x)|_{V^*}^2 + |x|^2)\mu_t^\alpha(dx)dt.$$

By this and similar arguments as in the proof of [BDR10, Theorem 2.6], (2.1) is verified and the assertion follows. $\square$

3 Application

Let $H = L^2(0,1) := L^2((0,1), d\xi)$, with $d\xi$ = Lebesgue measure, and let $A : D(A) \subset H \to H$ be defined by

$$Ax(\xi) = \frac{\partial^2}{\partial \xi^2} x(\xi), \xi \in (0,1), \quad D(A) = H^2(0,1) \cap H_0^1(0,1).$$

Then $V = H_0^1(0,1)$. Let $D(F) := [0,T] \times L^{2m}(0,1)$ and for $(t,x) \in D(F)$

$$F := F_1 + F_2, \quad F_1(t,x)(\xi) := f(\xi,t,x(\xi)) + h(\xi,t,x(\xi)), \quad F_2(t,x)(\xi) := \partial_\xi g(\xi,t,x(\xi)), \xi \in (0,1),$$

where F_2 takes values in $V^* := H^{-1}$. Here $f, h, g : (0,1) \times [0,T] \times \mathbb{R} \to \mathbb{R}$ are functions such that for every $\xi \in (0,1)$ the maps $f(\xi,\cdot,\cdot), h(\xi,\cdot,\cdot), g(\xi,\cdot,\cdot)$ are continuous on $(0,T) \times \mathbb{R}$ and satisfy the following conditions:

(f1) There exist $m \in \mathbb{N}$ and a nonnegative function $c_1 \in L^2(0,T)$ such that for all $t \in [0,T], z \in \mathbb{R}, \xi \in (0,1)$ one has

$$|f(\xi,t,z)| \le c_1(t)(1 + |z|^m).$$

(f2) There is a nonnegative function $c_2 \in L^1(0,T)$ such that for all $t \in [0,T], z_1, z_2 \in \mathbb{R}, \xi \in (0,1)$ one has

$$(f(\xi,t,z_1+z_2) - f(\xi,t,z_1))z_2 \le c_2(t)(|z_2|^2 + |z_1|^m + 1).$$

(h1) There exists a nonnegative function $c_3 \in L^2(0,T)$ such that for all $t \in [0,T], z \in \mathbb{R}, \xi \in (0,1)$, one has

$$|h(\xi,t,z)| \le c_3(t)(1+|z|).$$

(g1) The function g is of the form $g(\xi,t,r) = g_1(\xi,t,r) + g_2(t,r)$, where g_1 and g_2 are Borel functions of $(\xi,t,r) \in (0,1) \times [0,T] \times \mathbb{R}$ and of $(t,r) \in [0,T] \times \mathbb{R}$, respectively. The function g_1 satisfies a linear growth and the function g_2 a quadratic growth condition, i.e. there is a constant K such that

$$|g_1(\xi,t,r)| \le K(1+|r|), \quad |g_2(t,r)| \le K(1+|r|^2),$$

for all $t \in [0,T], \xi \in (0,1), r \in \mathbb{R}$.

(g2) g is a locally Lipschitz function with linearly growing Lipschitz constant, i.e. there exists a constant L such that

$$|g(\xi,t,p) - g(\xi,t,q)| \le L(1+|p|+|q|)|p-q|,$$

for all $t \in [0,T], \xi \in (0,1), p,q \in \mathbb{R}$.

For $\alpha \in (0,1]$ and $(t,x) \in [0,T] \times D(F)$ we define $F_\alpha : [0,T] \times D(F) \to V^*$,

$$F_\alpha := F_1^\alpha + F_2, \quad F_1^\alpha(t,x) := \frac{F_1(t,x)(\xi)}{1+\alpha|F_1(t,x)(\xi)|},$$

By Girsanov's Theorem (cf. [MR99, Theorem 3.1], [DFPR12, Theorem 13]) , we obtain that there exists a martingale solution for the following stochastic differential equation

$$dX_\alpha(t) = [AX_\alpha(t) + F_\alpha(t, X_\alpha(t))]dt + \sqrt{C}dW(t), X_\alpha(s) = x, s \le t, \tag{3.1}$$

for all $x \in H$. F_α has all properties in Hypothesis 2.2 (i), (iii).

Define for $m \ge 1$

$$J(t,x) := \begin{cases} 2(c_1(t) + c_3(t) + K)(1+|x|^m_{L^{2m}(0,1)}), & \text{if } (t,x) \in D(F) \\ +\infty, & \text{otherwise.} \end{cases}$$

$$|F_2(t,x)|_{V^*} \le K(1+|x|^2_{L^4}) \le J(t,x) < \infty \quad \forall (t,x) \in D(F) = [0,T] \times L^{2m}(0,1).$$

Then by (f1) and (h1) we obtain

$$|F(t,x)|_{V^*} \le J(t,x) < \infty \quad \forall (t,x) \in D(F) = [0,T] \times L^{2m}(0,1).$$

Proposition 3.1 For any $s \in [0,T)$, there exists $C \in (0,\infty)$, such that for $\alpha \in (0,1], x \in L^{2m}(0,1)$

$$E(|X_\alpha(t,s,x)|^{2m}_{L^{2m}(0,1)}) \le C(1+|x|^{2m}_{L^{2m}(0,1)}), \quad \forall t \in [s,T].$$

Proof Set $Y_\alpha(t) := X_\alpha(t,s,x) - W_A(s,t), t \in [s,T]$. Then we obtain

$$dY_\alpha(t) = [AY_\alpha(t) + F_\alpha(t, Y_\alpha(t) + W_A(s,t))]dt, Y_\alpha(s) = x, s \leq t. \tag{3.2}$$

Here the equation is again meant in the mild sense. Since the trajectories of W_A can be uniformly approximated, on any finite interval $[0,T]$, by functions W_A^n from $C([0,T], H_0^2)$ (e.g. such W_A^n can be obtained by taking convolutions with smooth functions), we can replace W_A by a smooth function W_A^n. By a standard method (see e.g. [GRZ09]) we obtain that there exists a weak solution $Y_\alpha^n \in L^\infty([s,T],H) \cap L^2([s,T],H_0^1)$ for (3.2) and we have for t_0 such that $t_0 - s$ is small enough

$$E\int_s^{t_0} \int |Y_\alpha^n(t)|^{2m-2} |\partial_\xi Y_\alpha^n(t)|^2 d\xi dt < C. \tag{3.3}$$

(This estimate can be obtained by taking bounded smooth function g_ε to approximate g and obtain the associated solution $Y_\alpha^{n,\varepsilon} \in L^\infty((s,T],H^1)$. Then by the same calculations as below we obtain (3.3) for $Y_\alpha^{n,\varepsilon}$ and letting $\varepsilon \to 0$ (3.3) follows.)

Now multiplying both sides of the equation by $(Y_\alpha^n(t))^{2m-1}$ and integrating with respect to $d\xi$ we obtain for $t \in [s,T]$

$$\begin{aligned}
&\frac{1}{2m}\frac{d}{dt}\int |Y_\alpha^n(t)|^{2m} d\xi + (2m-1)\int |Y_\alpha^n(t)|^{2m-2}|\partial_\xi Y_\alpha^n(t)|^2 d\xi \\
=&\int F_1^\alpha(t, Y_\alpha^n(t) + W_A^n(s,t)) Y_\alpha^n(t)^{2m-1} d\xi + {}_{V^*}\langle F_2(t, Y_\alpha^n(t) + W_A^n(s,t)), Y_\alpha^n(t)^{2m-1}\rangle_V \\
:=&I_1 + I_2.
\end{aligned}$$

Let us estimate I_2. We have

$$\begin{aligned}
&{}_{V^*}\langle F_2(t, Y_\alpha^n(t) + W_A^n(s,t)), Y_\alpha^n(t)^{2m-1}\rangle_V \\
=&{}_{V^*}\langle [F_2(t, Y_\alpha^n(t) + W_A^n(s,t)) - F_2(t, Y_\alpha^n(t))], Y_\alpha^n(t)^{2m-1}\rangle_V + {}_{V^*}\langle F_2(t, Y_\alpha^n(t)), Y_\alpha^n(t)^{2m-1}\rangle_V.
\end{aligned} \tag{3.4}$$

For the first term on the right hand side of (3.4), we have by (g2), and Young's inequality

$$\begin{aligned}
&{}_{V^*}\langle [F_2(t, Y_\alpha^n(t) + W_A^n(s,t)) - F_2(t, Y_\alpha^n(t))], Y_\alpha^n(t)^{2m-1}\rangle_V \\
\leq& C\int (1 + |Y_\alpha^n(t)| + |W_A^n(s,t)|)|W_A^n(s,t)||Y_\alpha^n(t)|^{2m-2}|\partial_\xi Y_\alpha^n(t)| d\xi \\
\leq& \frac{1}{2}\int |Y_\alpha^n(t)|^{2m-2}|\partial_\xi Y_\alpha^n(t)|^2 d\xi + C\int |W_A^n(s,t)|^2 |Y_\alpha^n(t)|^{2m} d\xi \\
&+ C\int (1 + |W_A^n(s,t)|)|W_A^n(s,t)||Y_\alpha^n(t)|^{2m-2}|\partial_\xi Y_\alpha^n(t)| d\xi \\
\leq& \int |Y_\alpha^n(t)|^{2m-2}|\partial_\xi Y_\alpha^n(t)|^2 d\xi + c|W_A^n(s,t)|_{L^{4m}}^{4m} + c|W_A^n(s,t)|_{L^{2m}}^{2m} + (c|W_A^n(s,t)|_{L^\infty}^2 + c)|Y_\alpha^n(t)|_{L^{2m}}^{2m}.
\end{aligned}$$

For the second term on the right hand side of (3.4), we have

$$\int_0^1 g_2(t, Y_\alpha^n) Y_\alpha^n(t)^{2m-2} \partial_\xi Y_\alpha^n(t) d\xi = \int_0^1 \partial_\xi g_3(t, Y_\alpha^n) d\xi = 0,$$

where $g_3(t,r)=\int_0^r g_2(t,z)z^{2m-2}dz$. Then we obtain

$$
\begin{aligned}
{}_{V^*}\langle F_2(t,Y_\alpha^n(t)),Y_\alpha^n(t)^{2m-1}\rangle_V=&-(2m-1)\int g_1(\xi,t,Y_\alpha^n)Y_\alpha^n(t)^{2m-2}\partial_\xi Y_\alpha^n(t)d\xi\\
\leq& C\int(1+|Y_\alpha^n(t)|)|Y_\alpha^n(t)|^{2m-2}|\partial_\xi Y_\alpha^n(t)|d\xi\\
\leq&\int(|Y_\alpha^n(t)|^{2m-2}|\partial_\xi Y_\alpha^n(t)|^2+C|Y_\alpha^n(t)|^{2m}+C)d\xi.
\end{aligned}
$$

I_1 we can estimate as in [BDR10]. Then we obtain

$$
\begin{aligned}
\frac{1}{2m}\frac{d}{dt}\int|Y_\alpha^n(t)|^{2m}d\xi\leq& c(t)\int[1+(1+\frac{2m-1}{2m})|Y_\alpha^n(t)|^{2m}+\frac{1}{2m}|W_A^n(s,t)|^{2m^2}+\frac{1}{m}|W_A^n(s,t)|^{m^2}d\xi\\
&+c|W_A^n(s,t)|_{L^{4m}}^{4m}+c|W_A^n(s,t)|_{L^{2m}}^{2m}+C+(c|W_A^n(t)|_{L^\infty}^2+c)|Y_\alpha^n(t)|_{L^{2m}}^{2m},
\end{aligned}
$$

where $c(t)=c_1+c_2+2c_3$. Then by Gronwall's lemma we have

$$
\begin{aligned}
|Y_\alpha^n(t)|_{L^{2m}}^{2m}\leq& e^{\int_s^t C|c(t')|+C(|W_A^n(s,t')|_{L^\infty}^2+1)dt'}(|x|_{L^{2m}}^{2m}+C\int_s^t(|c(t')|\\
&+|W_A^n(s,t')|_{L^{2m^2}}^{2m^2}+|W_A^n(s,t')|_{L^{m^2}}^{m^2}+|W_A^n(s,t')|_{L^{4m}}^{4m}+|W_A^n(s,t')|_{L^{2m}}^{2m}+1)dt'\\
\leq& e^{\int_s^t C|c(t')|+C(|W_A(s,t')|_{L^\infty}^2+1)dt'}(|x|_{L^{2m}}^{2m}+C\int_s^t(|c(t')|\\
&+|W_A(s,t')|_{L^{2m^2}}^{2m^2}+|W_A(s,t')|_{L^{m^2}}^{m^2}+|W_A(s,t')|_{L^{4m}}^{4m}+|W_A(s,t')|_{L^{2m}}^{2m}+1)dt'
\end{aligned}
\tag{3.5}
$$

Taking expectation we obtain for $s\leq t\leq t_0$ such that t_0-s small enough,

$$E|Y_\alpha^n(t)|_{L^{2m}}^{2m}\leq C|x|_{L^{2m}}^{2m}+C.$$

Moreover, we obtain

$$E|Y_\alpha^n(t)|_{W^{1,2}(s,t_0,H^{-\beta})}\leq C|x|_{L^{2m}}^{2m}+C.$$

$$E|Y_\alpha^n(t)|_{L^2(s,t_0,H^1)}\leq C|x|_{L^{2m}}^{2m}+C.$$

Thus we get Y_α^n in $L^2(s,t_0,H)$ is tight. Also W_A^n in $L^2(s,t_0,H)$ is tight. Therefore, we have $X_\alpha^n:=Y_\alpha^n+W_A^n$ is tight in $L^2(s,t_0,H)$. Then X_α^n converges to some variables X'_α in distribution. By a standard method (cf. [DZ92]) we obtain X'_α is a martingale solution of (3.1). Thus by the weak uniqueness of (3.1) we obtain X'_α has the same distribution as X_α. By this we have for $s\leq t\leq t_0$,

$$E|Y_\alpha(t)|_{L^{2m}(0,1)}^{2m}\leq C|x|_{L^{2m}(0,1)}^{2m}+C.$$

Moreover,

$$E|X_\alpha(t,s,x)|_{L^{2m}(0,1)}^{2m}\leq C|x|_{L^{2m}(0,1)}^{2m}+C.$$

In fact, by [EK86, Theorem 4.2] and the weak uniqueness of the martingale solution of (3.1) we obtain that the laws of the martingale solutions $X_\alpha(t,s,x)$ of (3.1) form a Markov process. We use $\mu_{s,t}^\alpha(s,dy)$ to denote the distribution of $X_\alpha(t,s,x)$. Then by the Markov property we have for $0\leq s\leq t_1\leq t_2\leq T$

$$\mu_{s,t_2}^\alpha(x,dz)=\int\mu_{s,t_1}^\alpha(x,dy)\mu_{t_1,t_2}^\alpha(y,dz).$$

By this we obtain by iteration for any $t \in [s,T]$

$$\int |z|_{L^{2m}}^{2m} \mu_{s,t}^{\alpha}(x,dz) = \int\int |z|_{L^{2m}}^{2m} \mu_{t_1,t}^{\alpha}(y,dz)\mu_{s,t_1}^{\alpha}(x,dy) \le C|x|_{L^{2m}(0,1)}^{2m} + C.$$

$\square$

Theorem 2.3 now applies to all $\zeta \in \mathcal{P}(H)$ such that

$$\int_H |x|_{L^{2m}(0,1)}^{2m} \zeta(dx) < \infty.$$

Remark 3.2 (i) Here we choose the L^{2m}-norm as a Lyapunov function J in Hypothesis 2.2. In [RS06], the first named author of this paper and Sobol studied the above semilinear stochastic partial differential equations with time independent coefficients. They also choose the L^{2m}-norm as a Lyapunov function with weakly compact level sets for the Kolmogorov operator L_0 and by analyzing the resolvent of the operator L they constructed the martingale solution to this problem if the noise is trace-class. In this paper, we concentrate on the space-time white noise and the method of constructing Lyapunov functions with weakly compact level sets for the Kolmogorov operator L_0, so far could not be used when $TrC = \infty$.

(ii) If $g \equiv 0$, we can obtain the uniqueness of the solution to the Fokker-Planck equation by [BDR11, Theorem 4.1].

4 Martingale solutions

In this section we consider the semilinear stochastic partial differential equation

$$dX(t) = (\frac{\partial^2}{\partial \xi^2}X(t) + f(t, X(t) + h(t, X(t)) + \frac{\partial}{\partial \xi}g(t, , X(t)))dt + \sqrt{C}dW(t), \tag{4.1}$$

with Dirichlet boundary condition

$$X(t,0) = X(t,1) = 0, t \in [0,T], \tag{4.2}$$

and the initial condition

$$X(0) = X_0 \in L^{2m}(0,1), \tag{4.3}$$

on H, where $f(t,\xi,r), h(t,\xi,r), g(t,\xi,r)$ are Borel measurable functions of $(t,\xi,r) \in \mathbb{R}_+ \times [0,1] \times \mathbb{R}$, W is a cylindrical Wiener process on H and $C \in L(H)$ is symmetric, nonnegative and such that $C^{-1} \in L(H)$. Here we assume that f, g, h satisfy the same conditions (f1),(f2),(h1),(g1),(g2) as in the previous section. However, additionally we have to assume that

$$c_1, c_3 \text{ is bounded}, \tag{4.4}$$

where c_1 is as in (f1) and c_3 in (h1). In this section we use a similar approximation as in Section 3 to get the existence of a martingale solution to (4.1), which also provides a solution to the associated Fokker-Planck equation. In fact, setting

$$P_{s,t}\varphi(x) = E[\varphi(X(t,s,x))], x \in L^{2m}(0,1) \quad 0 \le s < t \le T, \varphi \in \mathcal{B}_b(H),$$

and

$$\mu_t(dx) := (P_{s,t})^* \zeta(dx),$$

where $\zeta \in \mathcal{P}(L^{2m}(0,1))$ is the initial condition, at $t = s$. Then $\mu_t(dx)dt$ is a solution to the Fokker-Planck equation.

Theorem 4.1 Suppose that (f1), (f2), (h1) (g1), (g2) hold with bounded c_1, c_3 in (f1), (h1). For each initial value $x \in L^{2m}(0,1)$ there exists a martingale solution to problem (4.1)-(4.3), i.e. there exists a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$, a cylindrical Wiener process W on H and a progressively measurable process $X : [0,T] \times \Omega \to H$, such that for P-a.e. $\omega \in \Omega$,

$$X(\cdot, \omega) \in C([0,T]; L^{2m}(0,1))$$

and for all $\phi \in C^2([0,1])$

$$\begin{aligned}\langle X(t), \phi\rangle =& \langle x, \phi\rangle + \int_0^t \langle X(s), \phi''\rangle ds + \int_0^t \langle f(s, X(s)) + h(s, X(s)), \phi\rangle ds \\ &- \int_0^t \langle g(s, X(s)), \phi'\rangle ds + \int_0^t \langle \phi, \sqrt{C} dW(s)\rangle, \quad \forall t \in [0,T], \quad P - a.s..\end{aligned}$$

Moreover, if P, P' are two martingale solutions to problem (4.1)-(4.3) with the same initial value $x \in L^{2m}$ and

$$\int_0^t |X(s)|_{L^{2m}}^{2m} ds < \infty \ P + P' - a.s.,$$

for all $t > 0$, then $P = P'$.

Proof For f satisfying (f1), (f2) and h satisfying (h1) we choose

$$f_\alpha(t, \xi, r) = \frac{f(t, \xi, r)}{1 + \alpha |f(t, \xi, r)|}, h_\alpha(t, \xi, r) = \frac{h(t, \xi, r)}{1 + \alpha |h(t, \xi, r)|}.$$

Then by Girsanov's Theorem (cf. [MR99, Theorem 3.1], [DFPR12, Theorem 13]) there exists a martingale solution X_α to (4.1)-(4.3) with $f = f_\alpha, h = h_\alpha$. Then we obtain for almost every $\omega \in \Omega$ for all $t \in [0,T]$

$$\begin{aligned}X_\alpha(t) =& e^{tA} X_0 + \int_0^t e^{(t-s)A} f_\alpha(s, X_\alpha(s)) ds + \int_0^t e^{(t-s)A} h_\alpha(s, X_\alpha(s)) ds \\ &+ \int_0^t e^{(t-s)A} \partial_\xi g(s, X_\alpha(s)) ds + \int_0^t e^{(t-s)A} \sqrt{C} dW(s)\end{aligned} \tag{4.5}$$

for $d\xi$-almost every $\xi \in [0,1]$, where A is the same operator as in Section 3 and

$$\int_0^t e^{(t-s)A} \partial_\xi g(s, X_\alpha(s)) ds := -\int_0^t \int_0^1 \partial_y p_{t-s}(\xi, y) g(s, X_\alpha(s))(y) dy ds,$$

for the heat kernel $p_{t-s}(\xi, y)$ of $e^{(t-s)A}$. Define

$$W_A(t) := \int_0^t e^{(t-s)A} \sqrt{C} dW(s).$$

By [G98, Corollary 4.3] we can deduce that for every $p \geq 1, T > 0$

$$E(\sup_{(t,\xi)\in[0,T]\times[0,1]} |W_A(t,\xi)|^p) < \infty.$$

Hence $Y_\alpha := X_\alpha - W_A$ is a solution of the following equation:

$$Y_\alpha(t) = e^{tA}X_0 + \int_0^t e^{(t-s)A}(f_\alpha + h_\alpha)(s, Y_\alpha(s) + W_A(s))ds + \int_0^t \int_0^1 e^{(t-s)A}\partial_\xi g(s, Y_\alpha(s) + W_A(s))ds.$$

By [EK86, Theorem 4.2] and the weak uniqueness of the martingale solution of (4.5) we obtain that the laws of the solutions $X_\alpha(t,s,x)$ form a Markov process. By this we obtain that the laws of $Y_\alpha(t,s,x)$ also form a Markov process. We use $\nu^\alpha_{s,t}(x,dy)$ to denote the distribution of $Y_\alpha(t)$ with initial value x at time s. By the Markov property we have for $0 \leq s \leq t_1 \leq t_2 \leq T$

$$\nu^\alpha_{s,t_2}(x,dz) = \int \nu^\alpha_{s,t_1}(x,dy)\nu^\alpha_{t_1,t_2}(y,dz).$$

By the same argument as in the proof of Proposition 3.1, we obtain for $t - s \leq t_0$

$$\int_s^t \int |\partial_\xi(y^m)|^2_{L^2}\nu^\alpha_{s,t'}(x,dy)dt' \leq C|x|^{2m}_{L^{2m}} + C,$$

and

$$\int |y|^{2m}_{L^{2m}}\nu^\alpha_{s,t}(x,dy) \leq C|x|^{2m}_{L^{2m}} + C,$$

Then for $0 \leq s \leq T$, we have

$$\begin{aligned}
&\int_s^{2t_0+s} \int |\partial_\xi(z^m)|^2_{L^2}\nu^\alpha_{s,t'}(x,dz)dt' \\
\leq &\int_s^{s+t_0} \int |\partial_\xi(z^m)|^2_{L^2}\nu^\alpha_{s,t'}(x,dz)dt' + \int_{t_0+s}^{2t_0+s} \int\int |\partial_\xi(z^m)|^2_{L^2}\nu^\alpha_{s,s+t_0}(x,dy)\nu^\alpha_{s+t_0,t'}(y,dz)dt' \\
\leq &C|x|^{2m}_{L^{2m}} + C.
\end{aligned}$$

Then by iteration we obtain

$$E\int_0^T \int |Y_\alpha(t)|^{2m-2}|\partial_\xi Y_\alpha(t)|^2 d\xi dt < \infty.$$

Therefore, by the same arguments as in the proof of Proposition 3.1 we obtain the estimate (3.5) for Y_α with $0 \leq t \leq T$, which implies that $\sup_{t\leq T}|X_\alpha|^{2m}_{L^{2m}}$ is bounded in probability. Consider the sequences of the $L^2([0,1])$-valued stochastic processes $I^1_n(t)$ defined by

$$I^1_n(t) := \int_0^t e^{(t-s)A}(f_{\frac{1}{n}} + h_{\frac{1}{n}})(s, X_{\frac{1}{n}}(s))ds.$$

Since we have

$$\sup_{t\leq T}|f_\alpha(t,X_\alpha)|_{L^2} \leq \sup_{t\leq T} c_1(t)(1 + |X_\alpha|^m_{L^{2m}}),$$

and

$$\sup_{t\le T}|h_\alpha(t,X_\alpha)|_{L^2}\le \sup_{t\le T}c_3(t)(1+|X_\alpha|_{L^2}),$$

by [G98, Lemma 3.3] I_n^1 is tight in $E:=C([0,T];L^{2m}([0,1]))$, where we used the boundedness of c_1 and c_3. Similarly, $I_n^2(t):=\int_0^t e^{(t-s)A}\partial_\xi g(s,Y_{\frac{1}{n}}(s)+W_A(s))ds$ is tight in E. Furthermore for $x\in L^{2m}$, $I^0(t):=e^{(t-s)A}x, t\in[0,T]$, is in E, and the process

$$I^3(t):=\int_0^t e^{(t-s)A}\sqrt{C}dW(s),$$

is tight in $C([0,T]\times[0,1])$, where $C([0,T]\times[0,1])$ denotes the continuous function on $[0,T]\times[0,1]$. Therefore, the sequence of processes $X_n(t)=I^0(t)+I_n^1(t)+I_n^2(t)+I^3(t), t\in[0,T]$, is tight in E. Thus, by Skorokhod's representation theorem there exists a subsequences $n(k)$ and a sequence of random elements $\hat{X}_k, k=1,2,3,...$ in E, carried by some probability space $(\hat{\Omega},\hat{\mathcal{F}},\hat{P})$, such that $\hat{X}_k$ converges almost surely in E to a random element $\hat{X}$ for $k\to\infty$ and the distributions of $\hat{X}_k$ and $X_{\frac{1}{n_k}}$ coincide. Define

$$\begin{aligned}\hat{M}_k(\phi)(t):=&\langle\hat{X}_k(t)-x,\phi\rangle-\int_0^t\langle\hat{X}_k(s),A\phi\rangle ds-\int_0^t\langle(f_{1/n_k}+h_{1/n_k})(s,\hat{X}_k(s)),\phi\rangle ds\\&+\int_0^t\int_0^1\langle g(s,\hat{X}_k(s)),\phi'\rangle ds.\end{aligned}\tag{4.6}$$

$\hat{M}_k(\phi)$ is a family of martingales with respect to the filtration

$$\mathcal{G}_t^k=\sigma(\hat{X}_k(s),s\le t).$$

Taking $k\to\infty$ we have

$$\int_0^t\langle(f_{1/n_k}+h_{1/n_k})(s,\hat{X}_k(s)),\phi\rangle ds\to\int_0^t\langle(f+h)(s,\hat{X}(s)),\phi\rangle ds,$$

and

$$\int_0^t\int_0^1\langle g(s,\hat{X}_k(s)),\phi'\rangle ds\to\int_0^t\int_0^1\langle g(s,\hat{X}(s)),\phi'\rangle ds$$

by (g2). Moreover, by the method from [DZ92] and the martingale representation theorem in [O05, Theorem 2], the existence of the martingale solution follows. The weak uniqueness follows by [MR99, Theorem 3.3]. □

To obtain pathwise uniqueness, we additionally assume that f satisfies the following inequality:

$$|f(t,\xi,p)-f(t,\xi,q)|\le L(1+|p|^{m-1}+|q|^{m-1})|p-q|.\tag{4.7}$$

Theorem 4.2 Suppose that f satisfies (4.7). Then the solution of (4.1)-(4.3) is unique in $C([0,T];L^{2m}(0,1))$.

Proof Consider two solutions X_1, X_2 of (4.1)-(4.3) in the interval $[0,T]$. Then by [O04, Theorem 13] we obtain that X_1, X_2 also satisfy the mild equation, which implies that

$$X_1(t) - X_2(t) = \zeta_1(t) + \zeta_2(t),$$

where

$$\zeta_1(t) := \int_0^t e^{(t-s)A}(f(X_1)(s) - f(X_2)(s))ds,$$

$$\zeta_2(t) := \int_0^t e^{(t-s)A}\partial_\xi[g(X_1)(s) - g(X_2)(s)]ds,$$

For fixed ω, by [G98 Lemma 3.1] we obtain

$$\begin{aligned}|\zeta_1(t,\cdot)|_{L^{2m}} \leq& C\int_0^t (t-s)^{\frac{1}{4m}-1}|f(X_1)(s) - f(X_2)(s)|_{L^1}ds \\ \leq& C\int_0^t (t-s)^{\frac{1}{4m}-1}|X_1(s) - X_2(s)|_{L^{2m}}(1 + |X_1(s)|^{2m-1}_{L^{2m}} + |X_2(s)|^{2m-1}_{L^{2m}})ds \\ \leq& C(\sup_{0\leq s\leq T}|X_1(s)|_{L^{2m}}, \sup_{0\leq s\leq T}|X_2(s)|_{L^{2m}})\int_0^t (t-s)^{\frac{1}{4m}-1}|X_1(s) - X_2(s)|_{L^{2m}}ds.\end{aligned}$$

Similarly, we have

$$|\zeta_2(t,\cdot)|_{L^{2m}} \leq C(\sup_{0\leq s\leq T}|X_1(s)|_{L^{2m}}, \sup_{0\leq s\leq T}|X_2(s)|_{L^{2m}})\int_0^t (t-s)^{\frac{1}{4m}-1}|X_1(s) - X_2(s)|_{L^{2m}}ds.$$

Then we obtain

$$|X_1(t) - X_2(t)|_{L^{2m}} \leq C(\sup_{0\leq s\leq T}|X_1(s)|_{L^{2m}}, \sup_{0\leq s\leq T}|X_2(s)|_{L^{2m}})\int_0^t (t-s)^{\frac{1}{4m}-1}|X_1(s) - X_2(s)|_{L^{2m}}ds,$$

for every $t \in [0,T]$. Now the assertion follows by the Bellman-Gronwall lemma. $\square$

Combining Theorem 4.1 and Theorem 4.2 we obtain the following more general existence and uniqueness result by using the Yamada-Watanabe Theorem.

Theorem 4.3 Suppose that (f1), (f2), (4.7), (h1) (g1), (g2) hold with bounded c_1, c_3. Then for each initial condition $X_0 \in L^{2m}(0,1)$, there exists a pathwise unique probabilistically strong solution X of equation (4.1) over $[0,T]$ with initial condition $X(0) = X_0$, i.e. for every probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ with an $\mathcal{F}_t$-Wiener process W, there exists a unique $\mathcal{F}_t$-adapted process $X : [0,T] \times \Omega \to H$ such that for P-a.s. $\omega \in \Omega$

$$X(\cdot,\omega) \in C([0,T]; L^{2m}(0,1))$$

and for all $\phi \in C^2([0,1])$ we have P-a.s.

$$\begin{aligned}\langle X(t), \phi\rangle =& \langle X_0, \phi\rangle + \int_0^t \langle X(s), \phi''\rangle ds + \int_0^t \langle f(s, X(s)), \phi\rangle ds \\ &- \int_0^t \langle g(s, X(s)), \phi'\rangle ds + \int_0^t \langle \phi, dW(s)\rangle \quad \forall t \in [0,T].\end{aligned}$$

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